[2]Ardema, M.D. and Rajan, N., "Separation of Time-Scales in Aircraft Trajectory Optimization," *Journal of Guidance, Control, and Dynamics*, Vol. 8, March-April 1985, pp. 275-278.

[4]Rajan, N. and Ardema, M.D., "Interception in Three-Dimensions: An Energy Formulation," *Journal of Guidance, Control, and Dynamics*, Vol. 8, Jan.-Feb. 1985, pp. 23-30.

An Analytical Approach to Geosynchronous Station Acquisition

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Introduction

GENERALLY, the effect of any transfer from low Earth orbit to geosynchronous orbit will be to place a spacecraft into an orbit located some distance away from the desired longitude or "station" with drift relative to that station. Consequently, a series of relatively small impulsive maneuvers must be performed to move the vehicle to the desired location and to eliminate the drift. A number of techniques and software systems have been devised to study this problem.[1,2] One common technique, such as that employed in the analysis program used for the SMS and GOES spacecraft, is to systematically examine all possible combinations of maneuvers satisfying a set of user-specified constraints.[3] These different "acquisition maneuver designs" can then be sorted to select one or more meeting some desired criteria. While such systems have proved useful, the alternative approach discussed here allows the analyst considerably more insight and control over the maneuver design process.

Optimized Station Acquisition Maneuver Design

Suppose that as a consequence of the transfer to geosynchronous orbit, the spacecraft is located at a longitude λ_0 with a drift rate of $\dot{\lambda}_0$, both of which are different from the desired station location λ_s and zero drift rate. There are two characteristics of the station acquisition scheme that must be established: the minimum velocity change necessary to produce the desired orbital characteristics and the order in which to apply the impulses to rotate the longitude to that desired orbit.

The minimum velocity change necessary to produce the proper orbital conditions can be found using a coplanar Hohmann transfer.[4] This velocity change (ΔV_T), while establishing the desired orbital altitude and shape, will not rotate the longitude from one location to another. In order to accomplish this, the velocity change is split a number of times to determine a series of intermediate orbits. These orbits are chosen such that

$$\Delta\lambda - \epsilon \leq \sum_{i=0}^{N} R_i \dot{\lambda}_i \leq \Delta\lambda + \epsilon \tag{1}$$

where R_i is the number of days at each intermediate orbit i, $\dot{\lambda}_i$ the associated drift rate, $\Delta\lambda$ the longitude change desired, and ϵ a small tolerance about the desired station. Inspection of this result indicates that there seems to be no straightforward way of establishing an optimal set of impulsive burns.

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However, three basic constraints can put some limits upon the number of possible solutions. The first of these is a constraint on the maximum velocity change allowed for any one maneuver. This limit can be imposed by the nature of the spacecraft hardware (e.g., maximum thruster steady-state operation time) and avoidance of gravity losses incurred by excessively long maneuvers.[5] Another constraint is defined by the range of values allowed for the number of days in any of the intermediate orbits. The minimum value is nominally chosen to correspond to the least amount of time required to obtain adequate tracking data and to perform the processing necessary to determine the orbit of the satellite. This permits fine tuning of any subsequent maneuvers due to variations in thruster performance. The final constraint comes about by noting the ratio of required ΔV_T and the maximum ΔV allowed per maneuver. This value is the minimum number of burns needed to acquire station.

Having found the velocity change to effect the orbital transfer, a means is then necessary to estimate the minimum time to acquire the station. Noting that one part of the Hohmann transfer can initiate the drift to the station and that the second part can stop the drift once the station is reached, some coast time T between the two must be found. The desired drift rate to initiate the drift to station is simply $\Delta\lambda/T$. It can be noted that during the coast between maneuvers,

$$\lambda = \frac{1}{2}\left(\ddot{\lambda}_0 + \frac{1}{2}\Delta\lambda\frac{\partial\ddot{\lambda}}{\partial\lambda_0}\right)t^2 + \dot{\lambda}_0 t + \lambda_0 \tag{2}$$

This is a modification of the quadratic equations in Ref. 6 to reflect longitude variations greater than 1 deg. A key issue to this approach is the accuracy of this result. The direct numerical comparison of this equation and similar results from Ref. 7 for a 28 day epoch shows a 0.03 deg difference in longitude having started at the initial conditions of $\lambda = 272.1$ deg and $\dot{\lambda} = -1.472$ deg/day. As can be observed, the differences are small. Thus, it can be concluded that this relation can be used with confidence.

The relation between velocity change and drift rate change $[\Delta V = |\Delta\dot{\lambda}/n|V_s/3]$ can now be exploited. The first velocity change can be related to the longitude difference by

$$\Delta V_1 = \frac{V_s}{3n}\left[\frac{\Delta\lambda}{T} - \left(\ddot{\lambda}_0 + \frac{1}{2}\Delta\lambda\frac{\partial\ddot{\lambda}}{\partial\lambda_0}\right)T - \dot{\lambda}_0\right]$$

and the second given as

$$\Delta V_2 = \frac{V_s}{3n}\left[\frac{\Delta\lambda}{T} - \left(\ddot{\lambda}_0 + \frac{1}{2}\Delta\lambda\frac{\partial\ddot{\lambda}}{\partial\lambda_0}\right)T\right]$$

Table 1 Example orbit and maneuver specifications

Specification	Initial orbit	Final orbit
Semi-major axis, km	42,279.2	42,165
Eccentricity	0.017	0.0002
Longitude, deg	272.1	230.0
Drift rate, deg/day	−1.472	0.0
$\Delta V_T = 26.3$ m/s (Hohmann transfer), $\Delta V_{max} \approx 5$ m/s (pre maneuver)		
Maneuver no.	**ΔV m/s**	**Drift rate, deg/day**
0	–	−1.472
1	−5.0	−3.233
2	−5.0	−4.994
3	−1.2	−5.417
4	5.03	−3.642
5	5.03	−1.867
6	5.04	−0.092

where V_s is synchronous velocity and n the mean orbital motion. The first velocity impulse ΔV_1 initiates the drift to the desired station λ_s and accounts not only for the difference between the drift rates $\Delta\lambda/T$ and $\dot{\lambda}_0$, but also the motion induced by orbital mechanics. The second impulse ΔV_2 is needed to remove the residual drift rate and thus bring the satellite to rest at the station longitude. Since the value of $\Delta V_T (=\Delta V_1 + \Delta V_2)$ is known from the Hohmann transfer equations, the results above can be combined to yield the following identity:

$$2\left(\ddot{\lambda}_0+\frac{1}{2}\Delta\lambda\frac{\partial\ddot{\lambda}}{\partial\lambda_0}\right)T^2+\left(\dot{\lambda}_0+\frac{3n\Delta V_T}{V_s}\right)T-2\Delta\lambda=0 \qquad (3)$$

The smallest non-negative value from the solution to this quadratic equation is then chosen as the coast time T. As a result, the minimum time and the minimum velocity have now been found. However, what remains is the determination of the optimum set of maneuvers.

Generalization of the coast time equation (3) yields that of a parabola [i.e., $f(t)=at^2+bt+c$]. If the initial conditions at $t=0$ [i.e., $f(0)=0$] and at $t=T$ [i.e., $f(T)=\Delta\lambda$, $\dot{f}(T)=0$] are imposed, the following is found:

$$f(t)=(-\Delta\lambda/T^2)t^2+(2\Delta\lambda/T)t \qquad (4)$$

This equation describes the maneuver path as if there were a long continuous burn for time T from the initial location to the final location (i.e., station longitude λ_s). Thus, Eq. (3) can be considered as the two-burn approximation to the continuous burn strategy represented by Eq. (4).

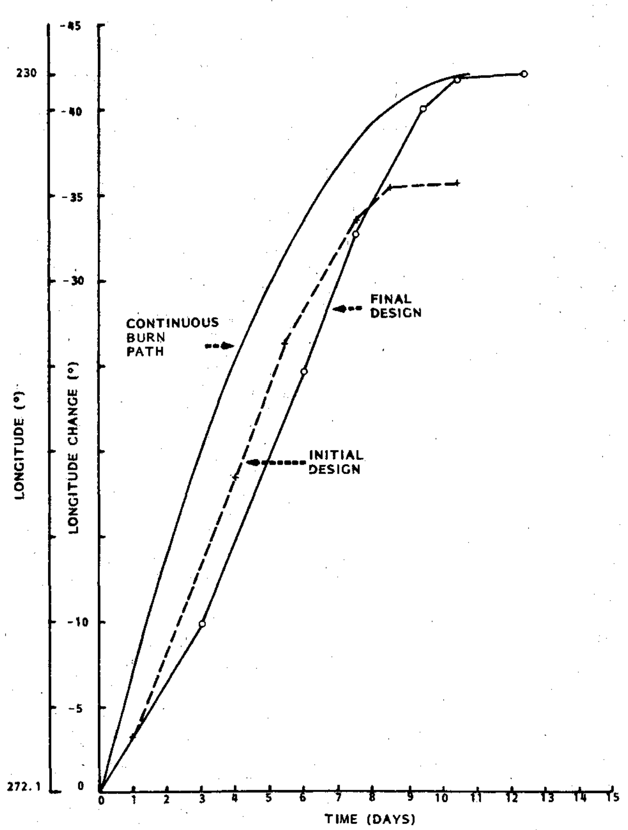

Fig. 1 Analytic maneuver designs.

Table 2 Initial maneuver design estimate

Burn no.	Calculated coast time, days	Selected coast time, days	Longitude change, deg
1	–	1	−3.23
2	3.1	3	−14.98
3	1.6	1.5	−8.13
4	1.6	2	−7.28
5	1.3	1	−1.87
6	1.9	2	−0.19
	Total	10.5	−35.68

Finite Burn Approximation

As there cannot be a continuous set of maneuvers, a finite burn approximation is necessary to the solution in Eq. (4). Manipulation of Eq. (2) and the definition of ΔV noted earlier can yield the change in longitude as a function of time due to an impulsive maneuver,

$$\delta\lambda=\left[\frac{3n\Delta V}{V_s}\Delta t+\dot{\lambda}_0\Delta t+\frac{1}{2}\ddot{\lambda}_0\Delta t^2\right]\left[1-\frac{1}{4}\left(\frac{\partial\ddot{\lambda}}{\partial\lambda_0}\right)\Delta t^2\right]^{-1}$$

Performing N such maneuvers will yield a function $g(t)$ that describes the longitude change produced by the maneuvers

$$g(t)=\sum_{K=1}^{N}\left\{\left[\left(\dot{\lambda}_{K-1}+\frac{3n\Delta V_K}{V_s}\right)\Delta t_K+\frac{1}{2}\ddot{\lambda}_{K-1}\Delta t_K^2\right]\left[1-\frac{1}{4}\frac{\partial\ddot{\lambda}}{\partial\lambda_{K-1}}\Delta t_K^2\right]^{-1}\right\} \qquad (5)$$

This equation is an approximation to that in Eq. (4). The velocity elements, ΔV_K, can be defined by the minimum number of burns and total velocity constraints or, alternatively, by some arbitrary selection. Therefore, the selection of the Δt_K's needs to be performed. This can be done by trial and error or, better yet, by trying to minimize the error between the two functions, i.e., Eqs.(4) and (5).

A classic approach for minimizing the error is to minimize the weighted sum of the squares of the deviations between the finite burn approximation and the exact solution.[8] Replacing continuous time with the summation of the discrete times Δt_K in Eq. (4) allows the sum of the squares of the deviations, nominally called "chi-squared," to be written as

$$\chi^2=\sum_{J=1}^{N}\left\{\sum_{K=1}^{J}\left[-\left(\frac{\Delta\lambda}{T^2}+\frac{1}{2}\ddot{\lambda}_{K-1}\right)\Delta t_K^2+\left(\frac{2\Delta\lambda}{T}-\dot{\lambda}_{K-1}-\frac{3n\Delta V_K}{V_s}\right)\Delta t_K\right]-\left(\frac{2\Delta\lambda}{T^2}\right)\sum_{K=1}^{J-1}\Delta t_K\sum_{I=K+1}^{J}\Delta t_I\right\}^2$$

having assumed that the statistical weight on each deviation is one. Minimizing the above expression with respect to time can be used to solve for the time step Δt_m, giving

$$\Delta t_m=\left[\sum_{K=1}^{m}\left(\frac{2\Delta\lambda}{T}-\dot{\lambda}_{K-1}-\frac{3n\Delta V_K}{V_s}\right)-2\sum_{K=1}^{m-1}\left(\frac{\Delta\lambda}{T^2}+\frac{1}{2}\ddot{\lambda}_{K-1}\right)\Delta t_K-\left(\frac{2\Delta\lambda}{T^2}\right)\sum_{K=1}^{m-2}\sum_{I=K+1}^{m-1}\Delta t_I\right]\left[2\frac{(m+1)\Delta\lambda}{T^2}+\ddot{\lambda}_{m-1}\right]^{-1}$$

$$m\geq 3 \qquad (6)$$

noting that, for $m=2$, the double sum is not present in the above expression.

Using the results from Eqs. (3) and (6) in conjunction with the aforementioned constraints (ΔV_T, longitude change, maximum ΔV per burn, minimum number of maneuvers) allows a unique solution to the station acquisition problem to be found. The user of this technique must use some care as the first time step Δt_l must be specified and the appropriate apsis location should be ensured for all maneuvers. Nonetheless, this approach allows for an analytical solution to a problem that nominally requires a large software system.

Example Maneuver Design

As an example consider the situation summarized in Table 1. Using the minimum coast-time relation [Eq. (3)], the value for drift acceleration $\ddot{\lambda}$ and acceleration variation $\partial\ddot{\lambda}/\partial\lambda$ (which can be calculated from equations in Ref. 7), in addition to the other available information yields the time to acquire station—10.86 days. Consequently, the equation describing the continuous burn strategy is

$$f(t) = 0.357t^2 - 7.748t \text{ deg}$$

where time t is measured in days.

The procedure used in calculating the various coast time intervals [Eq. (6)] can now be used. Table 2 shows the values calculated as a preliminary estimate of the maneuver sequence. Note that the initial selected coast time has been set to 1 day and that the values have been rounded off or modified to put the satellite at the correct point in the orbit. While the amount of time to reach the station nearly matches the ideal, the longitude change is far short of that needed. However, it is easily observed that the difference is nearly equal (within 0.6%) to an additional 2 days of coast at the first maneuver drift rate. Thus, with this additional time, a unique solution has been found, using minimum velocity change and minimum number of maneuvers to enable the acquisition of the desired station in a nearly optimal fashion. Figure 1 shows the comparison between the initial and final designs and the continuous burn optimal path.

Conclusion

The analytical technique developed in this Note permits the estimation of an optimal maneuver sequence without enlisting the aid of massive computer software systems. It does this by minimizing the deviation between a finite approximation and the exact optimal solution to the problem of station acquisition. The solution found uses the minimum velocity change and minimum number of maneuvers to make a fixed longitude change. The approach taken is simple and powerful, allowing the analyst considerable insight and control over the maneuver sequences. An example demonstrates the utility of the approach.

References

[1]Greene, R.H., "Early Bird Placement in a Stationary Orbit: Launch and Control System Maneuvers," *AIAA Progress in Astronautics and Aeronautics, Volume 19: Communication Satellite System Technology,* R.B. Marsten ed., New York, 1966, pp. 9-42.

[2]Lewin, J., "Ground-Control System for Satcom Satellites," *Journal of Spacecraft and Rockets,* Vol. 16, July-Aug. 1979, p. 275.

[3]*"PRELUDE Program Users Guide,"* Computer Sciences Corp. 1975 (unpublished).

[4]Kaplan, M.H., *Modern Spacecraft Dynamics and Control,* John Wiley & Sons, New York, 1976, pp. 82-87.

[5]"Space Technology Trends and Forecasts," *NASA Space Systems Technology Model,* NASA/OAST, Jan. 1984, p. 5-46.

[6]Chao, C.C., "Semiautonomous Stationkeeping of Geosynchronous Satellites," *Journal of Guidance, Control, and Dynamics,* Vol. 7, Jan.-Feb. 1984, p. 57.

[7]Kamel, A., Ekman, D., and Tibbits, R., "East-West Stationkeeping Requirements of Nearly Synchronous Satellites Due to Earth's Triaxiality and Luni-Solar Effects," *Celestial Mechanics,* Vol. 8, 1973, pp. 129-148.

[8]Benington, P.R., *Data Reduction and Error Analysis for the Physical Sciences,* McGraw-Hill Book Co., New York, 1969, pp. 101-102.

Equations of Attitude Motion for an N-Body Satellite with Moving Joints

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Nomenclature

F_λ –total external force on body λ

F'_λ = nongravitational external force on body λ

$F^H_{\lambda j}$ = interaction force on body λ transmitted through joint j

$\hat{g}_j$ = unit vector along rotation axis of joint

J_λ = the set of joints on body λ

m = total mass

m_λ = mass of body λ

r = number of rotational degrees of freedom

S = the set of bodies in the topological tree

T_λ = total external torque on body λ

T'_λ = nongravitational external torque on body λ

$T^C_{\lambda j}$ = gimbal constraint torque on body λ at joint j

$T^H_{\lambda j}$ = torque on body λ transmitted through joint j

$T^{SD}_{\lambda j}$ = spring-damper torque on body λ at joint j

γ = planet's gravitational constant

γ_i = angle of rotation about axis $\hat{g}_i$

ρ = planetocentric position vector of satellite composite c.m.

$\hat{\rho}$ = unit vector in direction of ρ

$\hat{\rho}_\lambda$ = planetocentric position vector of c.m. of body λ

Φ_λ = inertia dyadic of body λ about center of mass

ω_0 –angular velocity of the reference body

ω_λ = angular velocity of body λ

1 = unit dyadic

Introduction

THE equations of spacecraft attitude dynamics derived by Hooker and Margulies[1] and their subsequent modification by Hooker[2] are well known and widely used.[3,4] This formulation is appropriate for the motion of a system of N rigid bodies connected by dissipative elastic joints and subject to arbitrary external forces and torques. The derivation follows an Eulerian method which accounts for internal and external torques in a straightforward way. The axes of rotation of each joint in the system and internal torque laws about these axes may be prescribed. Constraint torques at the joints do not appear in the final equations of motion, but their magnitudes may be determined a posteriori.[2] Two restrictions are imposed on the system in this formulation; chains of connected bodies may not form closed loops, and joint positions must be fixed with respect to the bodies connected at the joint.

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